

# Some properties of F-harmonic maps

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**ABSTRACT.** In this note, we investigate estimates of the Morse index for  $F$ -harmonic maps into spheres, our results extend partially those obtained in ([14]) and ([15]) for harmonic and  $p$ -harmonic maps.

## 1. Introduction

Harmonic maps have been studied first by J. Eells and J.H.Sampson in the sixties and since then many works were done ( see [4], [9], [13], [16], [17], [21]) to cite a few of them. Extensions to notions of  $p$ -harmonic, biharmonic,  $F$ -harmonic and  $f$ -harmonic maps were introduced and similar research has been carried out (see [1], [2], [3], [5], [12], [15], [18], [20]). Harmonic maps were applied to broad areas in sciences and engineering including the robot mechanics ( see [6], [8] ).

The Morse index for harmonic maps,  $p$ -harmonic maps, as well as biharmonic maps, into a standard unit Euclidean sphere  $S^n$  has been widely considered ( see [12], [14], [15],).

In this paper for a  $C^2$ -function  $F : [0, +\infty[ \rightarrow [0, +\infty[$  such that  $F'(t) > 0$  on  $t \in ]0, +\infty[$ , we consider the Morse index for  $F$ -harmonic maps into spheres. Our results generalize partial estimates of the Morse index obtained in ([14]) and ([15]) for harmonic and  $p$ -harmonic maps.

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $m \geq 2$ ,  $S^n$  the unit  $n$ -dimensional Euclidean sphere with  $n \geq 2$  endowed with the canonical metric *can* induced by the inner product of  $R^{n+1}$ .

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2000 *Mathematics Subject Classification.* Primary 58E20, 53C43.

*Key words and phrases.* F-harmonic maps, Morse index.

For a  $C^1$ - application  $\phi : (M, g) \longrightarrow (S^n, can)$ , we define the  $F$ -energy functional by,

$$E_F(\phi) = \int_M F \left( \frac{|d\phi|^2}{2} \right) dv_g$$

where  $\frac{|d\phi|^2}{2}$  denotes the energy density given by

$$\frac{|d\phi|^2}{2} = \frac{1}{2} \sum_{i=1}^m |d\phi(e_i)|^2$$

and where  $\{e_i\}$  is an orthonormal basis on  $T_x M$  and  $dv_g$  is the Riemannian measure associated to  $g$  on  $M$ .

Let  $\phi^{-1}TS^n$  be the pullback vector fiber bundle of  $TS^n$ ,  $\Gamma(\phi^{-1}TS^n)$  the space of sections on  $\phi^{-1}TS^n$  and denote by  $\nabla^M$ ,  $\nabla^{S^n}$  and  $\tilde{\nabla}$  Levi-Civita connections on  $TM$ ,  $TS^n$  and  $\phi^{-1}TS^n$  respectively.  $\tilde{\nabla}$  is defined by

$$\tilde{\nabla}_X Y = \nabla_{\phi_* X}^{S^n} Y$$

where  $X \in TM$  and  $Y \in \Gamma(\phi^{-1}TS^n)$ .

Let  $v$  be a vector field on  $S^n$  and  $(\phi_t^v)_t$  the flow of diffeomorphisms induced by  $v$  on  $S^n$  i.e.

$$\phi_0^v = \phi, \quad \frac{d}{dt} \phi_t^v |_{t=0} = v.$$

The first variation formula of  $E_F(\phi)$  is given by

$$\begin{aligned} \frac{d}{dt} E_F(\phi_t) |_{t=0} &= \int_M F' \left( \frac{|d\phi_t|^2}{2} \right) \langle \nabla_{\partial_t} d\phi_t, d\phi_t \rangle |_{t=0} dv_g \\ &= - \int_M \langle v, \tau_F(\phi) \rangle dv_g \end{aligned}$$

where  $\tau_F(\phi) = \text{trace}_g \nabla \left( F' \left( \frac{|d\phi|^2}{2} \right) d\phi \right)$  denotes the Euler-Lagrange equation of the  $F$ -energy functional  $E_F$ . Remark that if  $|d\phi|_{\phi^{-1}TN}$  is constant then  $\phi$  is harmonic if and only if  $\phi$  is  $F$ -harmonic.

**DEFINITION 1.**  $\phi$  is called  $F$ -harmonic if and only if  $\tau_F(\phi) = 0$  i.e.  $\phi$  is a critical point of  $F$ -energy functional  $E_F$ .

The second variation of  $E_F$  is given as

$$\frac{d^2}{dt^2} E_F(\phi_t) |_{t=0} = \frac{d}{dt} \int_M \frac{d}{dt} F \left( \frac{|d\phi_t|^2}{2} \right) |_{t=0} dv_g$$

$$\begin{aligned}
&= \int_M \left[ F'' \left( \frac{|d\phi|^2}{2} \right) \langle \nabla v, d\phi_t \rangle^2 + F' \left( \frac{|d\phi|^2}{2} \right) |\nabla v|^2 \right] dv_g \\
&\quad - \int_M \left\langle \nabla_{\partial t} \frac{\partial \phi_t}{\partial t} \Big|_{t=0}, \text{trace}_g \nabla \left( F' \left( \frac{|d\phi|^2}{2} \right) d\phi \right) \right\rangle dv_g \\
&\quad - \int_M F' \left( \frac{|d\phi|^2}{2} \right) \sum_{i=1}^m \langle R^{S^n}(v, d\phi(e_i)) d\phi(e_i), v \rangle dv_g
\end{aligned}$$

and since  $\phi$  is  $F$ -harmonic,  $\tau_F(\phi) = 0$ , then

$$\begin{aligned}
&\frac{d^2}{dt^2} E_F(\phi_t) \Big|_{t=0} = \int_M F'' \left( \frac{|d\phi|^2}{2} \right) \langle \nabla v, d\phi \rangle^2 dv_g + \\
(1.1) \quad &\int_M F' \left( \frac{|d\phi|^2}{2} \right) \left[ |\nabla v|^2 - \sum_{i=1}^m \langle R^{S^n}(v, d\phi(e_i)) d\phi(e_i), v \rangle \right] dv_g.
\end{aligned}$$

Along this paper we consider variation in directions of vector fields of the subspace  $\mathcal{L}(\phi)$  of  $\Gamma(\phi^{-1}TS^n)$  defined by

$$\mathcal{L}(\phi) = \{ \bar{v} \circ \phi, v \in \mathbb{R}^{n+1} \}$$

where  $\bar{v}$  is a vector field on  $S^n$  given by  $\bar{v}(y) = v - \langle v, y \rangle y$  for any  $y \in S^n$ ; it is known that  $\bar{v}$  is a conformal vector field on  $S^n$ . Obviously, if  $\phi$  is not constant,  $\mathcal{L}(\phi)$  is of dimension  $n + 1$ .

## 2. Morse index for $F$ -harmonic application

For any vector field  $v$  on  $S^n$  along  $\phi$ , we associate the quadratic form

$$Q_\phi^F(v) = \frac{d^2}{dt^2} E_F(\phi_t) \Big|_{t=0}.$$

The Morse index of the  $F$ -harmonic map is defined as the positive integer

$$Ind_F(\phi) = \sup \{ \dim N, N \subset \Gamma(\phi) \text{ such that } Q_\phi^F(v) \text{ negative defined on } N \}$$

where  $N$  is a subspace of  $\Gamma(\phi)$ . The Morse index measures the degree of the instability of  $\phi$  which is called  $F$ -stable if  $Ind_F(\phi) = 0$ . Let also  $S_g^F(\phi)$  be the  $F$ -stress-energy tensor defined by

$$\begin{aligned}
(2.1) \quad S_g^F(\phi) &= F' \left( \frac{|d\phi|^2}{2} \right) |d\phi|^2 g - 2 \left( F' \left( \frac{|d\phi|^2}{2} \right) + F'' \left( \frac{|d\phi|^2}{2} \right) \frac{|d\phi|^2}{2} \right) \phi^* can.
\end{aligned}$$

For  $x \in M$ , we put

$$S_g^{o,F}(\phi) = \inf \{ S_g^F(\phi)(X, X), X \in T_x M \text{ such that } g(X, X) = 1 \}.$$

The tensor  $S_g^F(\phi)$  will be called positive ( resp. positive defined) at  $x$  if  $S_g^{o,F}(\phi) \geq 0$  (resp.  $S_g^{o,F}(\phi) > 0$ ).

REMARK 1.  $F(t) = \frac{1}{p} (2t)^{\frac{p}{2}}$ , with  $p \in [2, +\infty[$ ,  $S_g^p(\phi)$  is the stress-energy tensor introduced by Eells and Lemaire for  $p = 2$  ([9]) or El Soufi for  $p \geq 4$ , ([13]).

In this note we state the following result

THEOREM 1. *Let  $\phi$  be an  $F$ -harmonic map from a compact  $m$ -Riemannian manifold  $(M, g)$  ( $m \geq 2$ ) into the Euclidean sphere  $S^n$  ( $n \geq 2$ ). Suppose that the  $F$ -stress-energy tensor  $S_g^F(\phi)$  of  $\phi$  is positive defined. Then the Morse index of  $\phi$ ,  $\text{Ind}_F(\phi) \geq n + 1$ .*

PROOF. Let  $w = \bar{v} \circ \phi \in \mathcal{L}(\phi)$  and put  $\langle v, \phi \rangle = \phi_v$ . For any point  $x \in M$ , we denote respectively by  $w^T(x)$  and  $w^\perp(x)$  the tangential and normal components of the vector  $w(x)$  on the spaces  $d\phi(T_x M)$  and  $d\phi(T_x M)^\perp$ . Let also  $\{e_1, \dots, e_m\}$  an orthonormal basis of  $T_x M$  which diagonalizes  $\phi^*$  and such that  $\{d\phi(e_1), d\phi(e_2), \dots, d\phi(e_l)\}$  forms a basis of  $d\phi(T_x M)$ .

If  $\left( F' \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) \neq 0$  at the point  $x$ , then

$$|\bar{v}^T(x)|^2 = \sum_{i=1}^l |d\phi(e_i)|^{-2} \langle \bar{v}(x), d\phi(e_i) \rangle^2$$

on the other hand, for any  $i \leq l$ , we have

$$2 \left( F' \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) |d\phi(e_i)|^2 = |d\phi|^2 F' \left( \frac{|d\phi(x)|^2}{2} \right)$$

$$(2.2) \quad -S_g^F(\phi)(x)(e_i, e_i) \leq |d\phi|^2 F' \left( \frac{|d\phi(x)|^2}{2} \right) - S_g^{o,F}(\phi)(x)$$

so

$$\begin{aligned} & \left( |d\phi|^2 F' \left( \frac{|d\phi(x)|^2}{2} \right) - S_g^{o,F}(\phi)(x) \right) |\bar{v}^T(x)|^2 \geq \\ & 2 \left( F' \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) \sum_{i=1}^l \langle \bar{v}(x), d\phi(e_i) \rangle^2 \end{aligned}$$

and since,

$$\langle \bar{v}(x), d\phi(e_i) \rangle^2 = \langle v - \langle v, \phi \rangle \phi, d\phi(e_i) \rangle^2$$

$$= \langle v, d\phi(e_i) \rangle^2 = |d\phi_v(e_i)|^2$$

we get

$$\left( |d\phi|^2 F' \left( \frac{|d\phi(x)|^2}{2} \right) - S_g^{o,F}(\phi)(x) \right) |w^T(x)|^2 \geq 2 \left( F' \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) |d\phi_v(x)|^2.$$

Now, taking into account (2.2), we infer that

$$\begin{aligned} & 2 \left( F' \left( \frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) \right) |d\phi_v(x)|^2 - |d\phi|^2 F' \left( \frac{|d\phi(x)|^2}{2} \right) |\bar{v}|^2 \\ & \leq -|d\phi|^2 F' \left( \frac{|d\phi(x)|^2}{2} \right) |\bar{v}^N(x)| - S_g^{o,F}(\phi)(x) |\bar{v}^T(x)|^2 \\ (2.3) \quad & \leq -S_g^{o,F}(\phi)(x) |\bar{v}(x)|^2 \end{aligned}$$

Now the second variation writes as

$$\begin{aligned} & \frac{d^2}{dt^2} E_F(\phi_t) \big|_{t=0} = \int_M F'' \left( \frac{|d\phi|^2}{2} \right) \langle \nabla \bar{v}, d\phi \rangle^2 dv_g \\ & + \int_M F' \left( \frac{|d\phi|^2}{2} \right) [|\nabla \bar{v}|^2 - |d\phi|^2 |\bar{v}|^2 + |d\phi_v|^2] dv_g \end{aligned}$$

Consequently, we have

$$\begin{aligned} Q_\phi^F(v) &= 2 \int_M \left( \frac{|d\phi|^2}{2} F'' \left( \frac{|d\phi|^2}{2} \right) + F' \left( \frac{|d\phi|^2}{2} \right) \right) |d\phi_v|^2 dv_g \\ &\quad - \int_M F' \left( \frac{|d\phi|^2}{2} \right) |d\phi|^2 |\bar{v} \circ \phi|^2 dv_g \end{aligned}$$

and taking account of the inequality (2.3), we get that

$$Q_\phi^F(v) \leq -2 \int_M S_g^{0,F}(\phi) |\bar{v}|^2 dv_g.$$

Finally since  $S_g^{0,F}(\phi)$  is positive defined, it follows that  $Q_F$  is negative defined on  $\mathcal{L}(\phi)$ . Hence

$$Ind_F(\phi) \geq n + 1.$$

□

### 3. Morse index of particular $F$ -harmonic maps

**3.1. Stability of the identity map.** In this section we borrow ideas from [12] to show the stability of the identity map. Let  $(M, g)$  be a compact manifold and consider the identity  $I$  on  $M$  which is obviously  $F$ -harmonic, the second variation formula of  $I$  writes as

$$(3.1) \quad Q_I^F(v) = F''\left(\frac{m}{2}\right) \sum_{i=1}^m \int_M \langle \nabla_{e_i} v, e_i \rangle^2 dv_g + F'\left(\frac{m}{2}\right) \int_M [|\nabla v|^2 - Ric_M(v, v)] dv_g.$$

If  $L_v$  denotes the Lie derivative in the direction of  $v$ , the Yano's formula [22] leads to

$$(3.2) \quad \int_M [|\nabla v|^2 - Ric_M(v, v)] dv_g = \int_M \left[ \frac{1}{2} |L_v g|^2 - (div(v))^2 \right] dv_g.$$

Now if  $(e_i)_i$  is an orthonormal basis on  $M$  which diagonalizes  $L_v g$  we obtain as in [12] that

$$(3.3) \quad |L_v g|^2 \geq \frac{4}{m} (div(v))^2$$

therefore by (3.1), (3.2) and (3.3) we infer that

$$(3.4) \quad Q_I^F(v) \geq \frac{1}{m} \left( F''\left(\frac{m}{2}\right) + (2-m) F'\left(\frac{m}{2}\right) \right) \int_M div(v)^2 dv_g.$$

We deduce the following proposition:

**PROPOSITION 1.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $m \geq 3$ . Suppose that*

$$(3.5) \quad F''\left(\frac{m}{2}\right) + (2-m) F'\left(\frac{m}{2}\right) \geq 0.$$

*The identity map  $I$  on  $M$  is  $F$ -stable.*

**REMARK 2.**  $F(t) = \frac{1}{2-m} e^{(2-m)t} + C$ , where  $C \geq \frac{1}{m-2}$  is a constant, fulfills the condition (3.5).

**3.2. Morse index of the identity map.** Now we are interested by the identity map  $I$  on  $M$ . Let  $C$  and  $K$  denote the space of conformal vector fields and the space of Killing vector fields on  $M$  respectively.

**PROPOSITION 2.** *Let  $(M, g)$  be a compact  $m$ -dimensional manifold ( $m \geq 3$ ). Suppose that*

$$(3.6) \quad \frac{m-2}{m} F'\left(\frac{m}{2}\right) - F''\left(\frac{m}{2}\right) > 0$$

then  $Ind_F(I) \geq \dim(C/K)$ .

PROOF. Plugging (3.1) in (3.2), we get

$$(3.7) \quad \begin{aligned} Q_I^F(v) &= F''\left(\frac{m}{2}\right) \int_M \operatorname{div}(v)^2 dv_g + \\ &F'\left(\frac{m}{2}\right) \int_M \left[ \frac{1}{2} |L_v g|^2 - \operatorname{div}(v)^2 \right] dv_g \end{aligned}$$

and if  $v$  is a conformal vector field on  $M$  then ( see the proof of Theorem 2 in [15] )

$$(3.8) \quad L_v g = -\frac{2}{m} \operatorname{div}(v) g$$

where  $m = \dim(M)$ . So (3.7) becomes

$$Q_I^F(v) = \left( F''\left(\frac{m}{2}\right) + \frac{2-m}{m} F'\left(\frac{m}{2}\right) \right) \int_M \operatorname{div}(v)^2 dv_g.$$

If  $\frac{m-2}{m} F'\left(\frac{m}{2}\right) - F''\left(\frac{m}{2}\right) > 0$ , then

$$Q_I^F(v) \leq 0.$$

The equality holds if  $\operatorname{div}(v) = 0$  which means by (3.8) that  $v$  is a Killing vector field. Then on the quotient space  $C/K$ , we have

$$Q_I^F(v) < 0$$

i.e.

$$Ind_F(I) \geq \dim(C/K).$$

□

REMARK 3.  $F(t) = \frac{m}{m-2} e^{\frac{m-2}{m}t} + Ct$ , where  $C > 0$  is a constant, fulfills the condition (3.6).

**3.3. Morse index of the homothetic map.** Let  $\phi : (M, g) \rightarrow (N, h)$  be a homothetic map i.e.  $\phi^*h = k^2g$  where  $k \in \mathbb{R}$ . Clearly  $|d\phi|_h^2 = mk^2$ , where  $m = \dim(M)$ , in that case the  $F$ -tension  $\tau_F(\phi)$  is proportional to the mean curvature of  $\phi$  so  $\phi$  is  $F$ -harmonic if and only if  $\phi$  is minimal immersion.

PROPOSITION 3. Let  $\phi : (M, g) \rightarrow (N, h)$  be an  $F$ -harmonic homothetic map. Then we have

$$Ind_F(\phi) \geq Ind_F(I)$$

where  $I$  is the identity map of  $M$ .

PROOF. The second variation of  $\phi$  in direction of a vector field  $v$  reduces to

$$(3.9) \quad Q_\phi^F(v) = F'''(\frac{mk^2}{2}) \int_M \langle \nabla v, d\phi \rangle_{\phi^{-1}T_N}^2 dv_g \\ + F'(\frac{mk^2}{2}) \int_M \left[ |\nabla v|^2 - \sum_{i=1}^m \langle R^N(v, d\phi(e_i)) d\phi(e_i), v \rangle \right] dv_g$$

where  $\{e_i\}_{1 \leq i \leq m}$  is an orthonormal basis on  $M$ . Let  $\Gamma^T(\phi)$  the subspace of  $\Gamma(\phi^{-1}TN)$ , consisting of vector fields on  $N$  of the form  $d\phi(X)$  where  $X$  is a vector field on  $M$ . The restriction of  $Q_\phi^I$  to  $\Gamma^T(\phi)$ , where  $I$  is the identity map on  $M$ , is given by (see Lemma 2.5 [15])

$$(3.10) \quad Q_\phi^I(d\phi(X)) = k^2 Q_I^I(X).$$

As in [15] and since  $\nabla d\phi$  takes its value in the normal fiber bundle of  $N$ , we get

$$(3.11) \quad \langle \nabla_X d\phi(Y), d\phi(Z) \rangle = \langle (\nabla d\phi)(X, Y), Z \rangle + \langle d\phi(\nabla_X Y), d\phi(Z) \rangle \\ = k^2 \langle \nabla_X Y, Z \rangle.$$

Replacing (3.11) and (3.10) in (3.9) we deduce that

$$Q_\phi^F(d\phi(X)) = F'''(\frac{mk^2}{2}) k^2 \int_M \langle \nabla_{e_i} X, e_i \rangle^2 dv_g + F'(\frac{mk^2}{2}) k^2 Q_I^I(X) \\ = k^2 Q_I^F(X).$$

□

Propositions (2) and (3) lead to

COROLLARY 1. *Let  $\phi : (M, g) \rightarrow (N, h)$  be an  $F$ -harmonic homothetic map. Suppose that*

$$\frac{m-2}{m} F'(\frac{m}{2}) - F''(\frac{m}{2}) > 0$$

where  $m = \dim(M) \geq 3$ .

Then

$$Ind_F(\phi) \geq \dim(C/K).$$

We can deduce an estimation to the  $F$ -index of an homothetic  $F$ -harmonic from Theorem 1.

Consider  $\phi : (M, g) \rightarrow (S^n, can)$  an homothetic map i.e.  $\phi^*can = k^2g$ ,  $k \in \mathbb{R}$ ; where  $S^n$  denotes the unit Euclidean  $n$ -dimensional sphere



endowed with the canonical metric *can*. The  $F$ -stress-energy tensor given by (2.1) writes

$$\begin{aligned} S_g^F(\phi) &= F'\left(\frac{|d\phi|^2}{2}\right) |d\phi|^2 g - 2 \left( F'\left(\frac{|d\phi|^2}{2}\right) + F''\left(\frac{|d\phi|^2}{2}\right) \frac{|d\phi|^2}{2} \right) \frac{|d\phi|^2}{m} g \\ &= \left( \left(1 - \frac{2}{m}\right) F'\left(\frac{|d\phi|^2}{2}\right) - \frac{|d\phi|^2}{m} F''\left(\frac{|d\phi|^2}{2}\right) \right) |d\phi|^2 g. \end{aligned}$$

So  $S_g^F(\phi)$  will be positive defined if  $\left(1 - \frac{2}{m}\right) F'\left(\frac{|d\phi|^2}{2}\right) - \frac{|d\phi|^2}{m} F''\left(\frac{|d\phi|^2}{2}\right) > 0$ . As a consequence of Theorem 1, we have

**PROPOSITION 4.** *Let  $\phi$  be an homothetic  $F$ -harmonic map from a compact  $m$ -Riemannian manifold  $(M, g)$  ( $m \geq 3$ ) into the Euclidean sphere  $S^n$ . Suppose that*

$$(3.12) \quad \left(1 - \frac{2}{m}\right) F'\left(\frac{|d\phi|^2}{2}\right) - \frac{|d\phi|^2}{m} F''\left(\frac{|d\phi|^2}{2}\right) > 0.$$

*Then the Morse index of  $\phi$ ,  $Ind_F(\phi) \geq n + 1$ .*

**REMARK 4.** *The function  $F(t) = \frac{m^2}{m-2} e^{\frac{m-2}{m^2}t}$ , with  $m \geq 3$  fulfills the condition (3.12) for homothetic maps  $\phi : (M, g) \rightarrow (S^n, can)$  i.e.  $\phi^*can = k^2g$  provided that  $k^2 < m$ .*

**REMARK 5.** *The space  $C$  of conformal vector fields on the unit Euclidean sphere  $S^n$  is of dimension  $\frac{1}{2}(n+1)(n+2)$  and that of Killing vector fields  $K$  is of dimension  $\frac{1}{2}n(n+1)$ . Then  $\dim(C/K) = n + 1$ . So we recover the result given by Corollary 1.*

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